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ON THE SUMMATION OF CERTAIN LEGENDRE SERIES

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SUMMARY

The purpose of this paper is to find some recurrence relations for sums of infinite series of the form $\Sigma\, P_\ell^n(\cos\theta)t^{\ell+m}/(\ell+m)$. This is achieved by transforming the sums into integrals and then using the recurrence relations for these integrals.

1. *Introduction*

Recently, while discussing the problem of deformation of an elastic sphere by internal displacement dislocations, we [2] encountered infinite series of the form $\Sigma P_\ell^n(\cos\theta)t^{\ell+m}/(\ell+m)$. Similar series also occur in problems of deformation of an elastic sphere by certain stress distributions over its surface, (e.g. [5]). The purpose of this paper is to find some recurrence relations for sums of such series. This is achieved by transforming the sums into integrals and then using the recurrence relations for these integrals. The derived recurrence relations can be used to sum these infinite series for all positive integral values of m and n. Ben-Menahem [1] has found similar recurrence relations for Legendre series with general term $P_\ell(\cos\theta)t^{\ell+m+1}/[(\ell+1).(\ell+2)\ldots(\ell+m+1)]$. These Legendre series can also be summed by using the results derived in the present paper.

Recurrence relations for the Legendre series $\Sigma P_\ell^n(\cos\theta)t^{\ell-m}/(\ell-m)$ and $\Sigma \dfrac{\partial P_\ell^n(\cos\theta)}{\partial\theta}\, t^{\ell\pm m}/(\ell\pm m)$ are also derived.

2. *Derivation of the recurrence relations*

We define

$$S_{m,n} = \sum_{\ell=n}^{\infty} \frac{t^{\ell+m}}{\ell+m}\, P_\ell^n(\cos\theta), \tag{1}$$

where $|t| \leqslant 1$, m and n are arbitrary positive integers and $P_\ell^n(\cos\theta)$ is the associated Legendre function of the first kind with the definition [7]

$$P_\ell^n(\cos\theta) = (\sin\theta)^n\, \frac{d^n P_\ell(\cos\theta)}{d(\cos\theta)^n} \tag{2}$$

and where $P_\ell(\cos\theta)$ is the Legendre polynomial of degree ℓ.
We know that

$$\sum_{\ell=0}^{\infty} P_\ell(\cos\theta)t^\ell = \frac{1}{s}\ , \tag{3}$$
$$(|t| \leqslant 1,\ 0 < \theta < \pi)$$

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where

$$s = (1-2t\cos\theta + t^2)^{\frac{1}{2}}. \tag{4}$$

From equations (2) to (4), we obtain

$$\sum_{\ell=n}^{\infty} P_\ell^n(\cos\theta)t^\ell = \frac{(2n)!}{n!\ 2^n}\ \frac{(\sin\theta)^n\ t^n}{s^{2n+1}} \tag{5}$$

and so

$$\sum_{\ell=n}^{\infty} P_\ell^n(\cos\theta)t^{\ell+m-1} = \frac{(2n)!}{n!\ 2^n}\ \frac{(\sin\theta)^n\ t^{n+m-1}}{s^{2n+1}}. \tag{6}$$

Integrating both sides of equation (6) with respect to t, we find

$$S_{m,n} = \sum_{\ell=n}^{\infty} \frac{t^{\ell+m}}{\ell+m} P_\ell^n(\cos\theta) = \frac{(2n)!}{n!\ 2^n}(\sin\theta)^n \int_0^t \frac{t^{n+m-1}dt}{s^{2n+1}}. \qquad (n+m \geqslant 1) \tag{7}$$

When m=n=0, we define

$$S_{0,0} = \sum_{\ell=1}^{\infty} \frac{t^\ell}{\ell} P_\ell(\cos\theta). \tag{8}$$

If we write

$$J_{m,n} = \int_0^t \frac{t^m dt}{s^{2n+1}} \tag{9}$$

then equation (7) can be written as

$$S_{m,n} = \frac{(2n)!}{n!\ 2^n}(\sin\theta)^n\ J_{n+m-1,n}\ . \qquad (n+m \geqslant 1) \tag{10}$$

Next we make use of the well-known relations [3, 4]

$$J_{m,n} = \cos\theta\,\frac{2n-2m+1}{2n-m}J_{m-1,n} + \frac{m-1}{2n-m}J_{m-2,n} - \frac{t^{m-1}}{(2n-m)s^{2n-1}} + \frac{\delta_{1m}}{2n-1}, \qquad (m\neq 2n,\ m \geqslant 1) \tag{11}$$

$$J_{2n,n} = \cos\theta\ J_{2n-1,n} + J_{2n-2,n-1} - \frac{t^{2n-1}}{(2n-1)s^{2n-1}}, \qquad (n \geqslant 1) \tag{12}$$

$$\begin{aligned} J_{0,n} &= \int_0^t \frac{dt}{s^{2n+1}} \\ &= \frac{t-\cos\theta}{(2n-1)(\sin\theta)^2 s^{2n-1}}\left\{1 + \sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\cdot\left(\frac{s}{\sin\theta}\right)^{2k}\right\} \\ &+ \frac{\cos\theta}{(2n-1)(\sin\theta)^2}\left\{1 + \sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\cdot\frac{1}{(\sin\theta)^{2k}}\right\}, \end{aligned} \qquad (n \geqslant 1) \tag{13}$$

$$J_{0,0} = \int_0^t \frac{dt}{s}$$

$$= \log\left\{\frac{s-\cos\theta+t}{1-\cos\theta}\right\} = \log\left\{\frac{1+\cos\theta}{s+\cos\theta-t}\right\} . \tag{14}$$

From equations (10) to (14), we arrive at the following recurrence relations: ($t \leq 1$, $0 < \theta < \pi$)

$$S_{m+1,n} = \cos\theta \frac{2m-1}{m-n} S_{m,n} - \frac{m+n-1}{m-n} S_{m-1,n}$$

$$+\frac{(2n)!\ (\sin\theta)^n}{n!\ 2^n}\left\{\frac{t^{m+n-1}}{(m-n)s^{2n-1}} + \frac{\delta_{(1-n)m}}{2n-1}\right\}, \tag{15}$$

$(m \neq n,\ m+n \geqslant 1)$

$$S_{n+1,n} = \cos\theta\, S_{n,n} + (2n-1)\sin\theta\, S_{n,n-1} - \frac{(2n-2)!}{(n-1)!\ 2^{n-1}}(\sin\theta)^n\left(\frac{t}{s}\right)^{2n-1}, \tag{16}$$

$(n \geqslant 1)$

$$S_{1-n,n} = \frac{(2n)!}{n!\ 2^n}(\sin\theta)^n J_{0,n} \tag{17}$$

$(n \geqslant 1)$

$$S_{1,0} = \log\left\{\frac{s-\cos\theta+t}{1-\cos\theta}\right\} = \log\left\{\frac{1+\cos\theta}{s+\cos\theta-t}\right\}. \tag{18}$$

The recurrence relations (15) and (16) together with equations (17) and (18) enable one to evaluate the infinite sums $S_{m,n} = \sum_{\ell=n}^{\infty} P_\ell^n(\cos\theta)t^{\ell+m}/(\ell+m)$ for all $|t| \leqslant 1$ and for all positive integral values of m and n, including zero, excluding the case m=n=0. To evaluate $S_{0,0}$, as defined in equation (8), we proceed as follows:
From equation (3) we have

$$\sum_{\ell=1}^{\infty} P_\ell(\cos\theta)t^{\ell-1} = \frac{1}{ts} - \frac{1}{t} . \tag{19}$$

Integrating, we get

$$S_{0,0} = \sum_{\ell=1}^{\infty}\frac{t^\ell}{\ell} P_\ell(\cos\theta) = \int_0^t \frac{1-s}{st}\, dt$$

$$= -\log\left\{\frac{1+s-t\cos\theta}{2}\right\}. \tag{20}$$

To these, we add another relation of the same type which is sometimes required, namely,

$$\sum_{\ell=n}^{\infty}(2\ell+1)t^\ell P_\ell^n(\cos\theta) = \frac{(2n+1)!}{n!\ 2^n}\ \frac{(\sin\theta)^n}{s^{2n+3}}\ t^n(1-t^2). \tag{21}$$

$(|t| \leqslant 1,\ 0 < \theta < \pi)$

This can be easily proved by a differentation of equation (5) with respect

to t followed by simple algebraic operations.
We consider next the sum

$$S'_{m,n} = \sum_{\ell=m+1}^{\infty} \frac{t^{\ell-m}}{\ell-m} P_\ell^n(\cos\theta)$$

$$= \sum_{\kappa=1}^{\infty} \frac{t^\kappa}{k} P^n_{\kappa+m}(\cos\theta). \qquad (|t| \leqslant 1) \tag{22}$$

Using the following recurrence relation [6] for the Legendre functions [with the necessary change due to different definition of $P_\nu^\mu(x)$]

$$P^\mu_{\nu+1}(x) = x\, P_\nu^\mu(x) + (\nu+\mu)\sqrt{1-x^2}\; P_\nu^{\mu-1}(x)\,, \tag{23}$$

equation (22) can be reduced to

$$S'_{m,n} = \cos\theta\, S'_{m-1,n} + (m+n-1)\sin\theta\, S'_{m-1,n-1}$$

$$+ \sin\theta \sum_{\kappa=1}^{\infty} t^k\, P^{n-1}_{m+\kappa-1}(\cos\theta). \qquad (n \geqslant 1,\ m \geqslant 1) \tag{24}$$

But from equation (5),

$$\sum_{\kappa=1}^{\infty} t^\kappa\, P^{n-1}_{m+\kappa-1}(\cos\theta) = \frac{(2n-2)!}{(n-1)!\,2^{n-1}} \frac{(\sin\theta)^{n-1}\, t^{n-m}}{s^{2n-1}}$$

$$- \sum_{\ell=0}^{\infty} t^{\ell-m+1} P_\ell^{n-1}(\cos\theta). \qquad (|t| \leqslant 1,\ 0 < \theta < \pi) \tag{25}$$

From equations (24) and (25), we obtain

$$S'_{m,n} = \cos\theta\, S'_{m-1,n} + (m+n-1)\sin\theta\, S'_{m-1,n-1}$$

$$+ \frac{(2n-2)!}{(n-1)!\;2^{n-1}} \frac{(\sin\theta)^n\, t^{n-m}}{s^{2n-1}} - \sin\theta \sum_{\ell=0}^{m-1} t^{\ell-m+1} P_\ell^{n-1}(\cos\theta). \qquad (n \geqslant 1,\ m \geqslant 1,\ |t| \leqslant 1,\ 0 < \theta < \pi) \tag{26}$$

Comparing equations (1) and (8) with (22), it is obvious that

$$S'_{0,n} = S_{0,n} \tag{27}$$

and so $S'_{0,n}$ can be found from equation (15) to (18) and (20). Further

$$S'_{m,0} = \sum_{\ell=m+1}^{\infty} \frac{t^{\ell-m}}{\ell-m} P_\ell(\cos\theta) \tag{28}$$

$$= \sum_{k=1}^{\infty} \frac{t^\kappa}{k} P_{\kappa+m}(\cos\theta). \tag{29}$$

Making use of the relation [6]

$$nP_n(x) = (2n-1)\, x\, P_{n-1}(x) - (n-1)P_{n-2}(x), \tag{30}$$

equation (29) reduces to

$$S'_{m,0} = \frac{2m-1}{m} \cos\theta \; S'_{m-1,0} - \frac{m-1}{m} S'_{m-2,0}$$
$$- \frac{1}{m\, t^m} \left\{ \sum_{\ell=m-1}^{\infty} \frac{t^{\ell+2}}{\ell+2} P_\ell(\cos\theta) - \cos\theta . \sum_{\ell=m}^{\infty} \frac{t^{\ell+1}}{\ell+1} P_\ell(\cos\theta) \right\}. \qquad (31)$$
$$(m \geqslant 1)$$

Using equation (1), (15) to (18), this becomes

$$S'_{m,0} = \frac{2m-1}{m} \cos\theta \; S'_{m-1,0} - \frac{m-1}{m} S'_{m-2,0}$$
$$- \frac{1}{m\, t^m} \left\{ s-1+\cos\theta . \sum_{\ell=0}^{m-1} \frac{t^{\ell+1}}{\ell+1} P_\ell(\cos\theta) - \sum_{\ell=0}^{m-2} \frac{t^{\ell+2}}{\ell+2} P_\ell(\cos\theta) \right\}. \qquad (32)$$
$$(m \geqslant 1, \; |t| \leqslant 1, \; 0 < \theta < \pi)$$

Further, from equations (20) and (27), we have

$$S'_{0,0} = -\log \left\{ \frac{1+s-t\cos\theta}{2} \right\}. \qquad (33)$$

From equations (32) and (33), one can evaluate $S'_{m,0}$ for all integral values of m. Therefore, from equations (26), (27), (32) and (33), $S'_{m,n}$ can be evaluated for all $|t| \leqslant 1$ and for all positive integral values of m and n including zero.

Lastly, let us consider the infinite sum

$$D_{m,n} = \sum_{\ell=n}^{\infty} \frac{t^{\ell+m}}{\ell+m} \frac{\partial P_\ell^n(\cos\theta)}{\partial\theta}, \qquad (34)$$
$$(m+n \geqslant 1)$$

$$D_{0,0} = \sum_{\ell=1}^{\infty} \frac{t^\ell}{\ell} \frac{\partial P_\ell(\cos\theta)}{\partial\theta}. \qquad (35)$$

Using equations (1), (5) and the relation [6]

$$(1-x^2) \frac{d\, P_\nu^\mu(x)}{dx} = -\nu x\, P_\nu^\mu(x) + (\nu+\mu)\, P_{\nu-1}^\mu(x), \qquad (36)$$

it follows at once that

$$\sin\theta\, D_{m,n} = (m-n)\, S_{m+1,n} - m\cos\theta\, S_{m,n}$$
$$+ \frac{(2n)!}{n!\; 2^n} (\sin\theta)^n \frac{(\cos\theta - t)\, t^{n+m}}{s^{2n+1}}.$$
$$(m+n \geqslant 1) \qquad (37)$$

From equation (20),

$$\sin\theta\, D_{0,0} = -\cos\theta + \frac{1}{s}(\cos\theta - t). \qquad (38)$$

From equations (37) and (38) and the recurrence relations for $S_{m,n}$, one can find $D_{m,n}$ for all positive integral values of m and n including zero.

Similarly, if

$$D'_{m,n} = \sum_{\ell=m+1}^{\infty} \frac{t^{\ell-m}}{\ell-m} \frac{\partial P_\ell^n(\cos\theta)}{\partial\theta}, \qquad (m \geq 1) \tag{39}$$

it can be proved that

$$\sin\theta\, D'_{m,n} = m\cos\theta\, S'_{m,n} - (m+n)S'_{m-1,n} - t\,P_m^n(\cos\theta) + \frac{(2n)!}{n!\,2^n}(\cos\theta - t)t^{n-m}\frac{(\sin\theta)^n}{s^{2n+1}} - (\cos\theta - t)\sum_{\ell=0}^{m} t^{\ell-m}P_\ell^n(\cos\theta), \qquad (m \geq 1) \tag{40}$$

$$D'_{0,n} = D_{0,n}. \tag{41}$$

Sums involving higher derivatives of Legendre functions can be handled similarly.

3. *Some explicit results*

For ready reference, we give below the values of $S_{m,n}$ for m=0, 1, 2, 3 and n=0, 1, 2:

$$\left.\begin{aligned} &S_{0,0}\ \text{[Equation (20)]}\\ &S_{1,0}\ \text{[Equation (18)]}\\ &S_{2,0} = s-1 + x\log\left\{\frac{s-x+t}{1-x}\right\}\\ &S_{3,0} = \tfrac{1}{2}st - \tfrac{3}{2}x(1-s) + \tfrac{1}{2}(3x^2-1)\log\left\{\frac{s-x+t}{1-x}\right\}\end{aligned}\right| \qquad (x = \cos\theta) \tag{42}$$

In general

$$S_{m+1,0} = J_{m,0} = \int_0^t \frac{t^m}{s}\,dt \tag{43}$$

This integral has been considered by Ben-Menahem [1] in some detail. There it has been shown that

$$\int \frac{t^m}{s}\,dt = h_m(t,x)s + P_m(x)\int\frac{dt}{s}, \tag{44}$$

where

$$h_m(t,x) = W_{m-1}(x)\sum_{\kappa=0}^{m-1} t^\kappa P_\kappa(x) - P_m(x)\sum_{\kappa=0}^{m-1} t^\kappa W_{\kappa-1}(x), \tag{45}$$

$$W_{m-1}(x) = \sum_{j=1}^{m} j^{-1} P_{j-1}(x)\,P_{m-j}(x). \tag{46}$$

From equations (14), (43) and (44), we have

$$S_{m+1,0} = h_m(t,x)s - h_m(0,x) + P_m(x)\log\left\{\frac{s-x+t}{1-x}\right\}. \qquad (m \geq 0) \tag{47}$$

$$
\left.\begin{aligned}
yS_{0,1} &= x+(t-x)/s \\
yS_{1,1} &= 1-(1-xt)/s \\
yS_{2,1} &= x-\frac{1}{s}[x+(1-2x^2)t] + (1-x^2)\log\left\{\frac{s-x+t}{1-x}\right\} \\
yS_{3,1} &= -2+3x^2+\frac{1}{s}[2-3x^2+xt(6x^2-5)+t^2(1-x^2)]+3x(1-x^2)\log\left\{\frac{s-x+t}{1-x}\right\}
\end{aligned}\right| \qquad (48)
$$

$$
\left.\begin{aligned}
S_{0,2} &= \frac{1+x^2}{1-x^2} + \frac{2x(t-x)}{s(1-x^2)} - \frac{1-xt}{s^3} \\
S_{1,2} &= \frac{2x}{1-x^2} + \frac{(t-x)(1+x^2)}{s(1-x^2)} + \frac{(2x^2-1)t-x}{s^3} \\
S_{2,2} &= \frac{2}{1-x^2} + \frac{xt(3-x^2)-2x^4+3x^2-3}{s(1-x^2)} + \frac{1}{s^3}[xt(4x^2-3)+1-2x^2] \\
S_{3,2} &= \frac{x(5-3x^2)}{1-x^2} - 2\,\frac{t(4x^4-7x^2+2)+x(2x^4-5x^2+4)}{s(1-x^2)} \\
&\quad + \frac{1}{s^3}[t(8x^4-8x^2+1)-x(4x^2-3)]+3(1-x^2)\log\left\{\frac{s-x+t}{1-x}\right\}
\end{aligned}\right| \qquad (49)
$$

where

$$
x = \cos\theta, \quad y = \sin\theta = \sqrt{1-x^2}, \qquad |t| \leqslant 1, \quad 0 < \theta < \pi. \qquad (50)
$$

Further

$$
\begin{aligned}
S'_{1,0} &= -x + \frac{1-s}{t} - x\log\left\{\frac{1+s-tx}{2}\right\} \\
yS'_{1,1} &= 2x^2-1 + \frac{1}{s}\left\{1-2x^2+xt\right\} - (1-x^2)\log\left\{\frac{1+s-tx}{2}\right\} \qquad (51) \\
S'_{1,2} &= x\,\frac{3-x^2}{1-x^2} + \frac{t-x}{s}\left\{\frac{2}{1-x^2} + \frac{1}{s^2}\right\}
\end{aligned}
$$

4. *Remarks*

We may add here that by using the generating function [6]

$$
\frac{1}{(1-2\alpha t+\alpha^2)^{\nu}} = \sum_{n=0}^{\infty} C_n^{\nu}(t)\alpha^n , \qquad (52)
$$

for the Gegenbauer polynomials and following the method applied in this paper, one can find recurrence relations for series involving Gegenbauer polynomials similar to those obtained in the present paper for sums of Legendre functions.

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REFERENCES

1. Ben-Menahem, A. — Summation of Certain Legendre Series and Related Difference Equations, J.Math. and Phys., 45, 1966.

2. Ben-Menahem, A. and Singh, S.J. — Eigenvector Expansions of Green's Dyads with Applications to Geophysical Theory, Geophys.J.Roy.Astron.Soc. (in press).

3. Bois, G.P. — Tables of Indefinite Integrals, Dover Publications, Inc., New York, 1961, pp.57-71, 102-104.

4. Ryshik, I.M. and Gradstein, I.S. — Tables of Series, Products and Integrals, Veb Deutscher Verlag Der Wissenschaften, Berlin, 1957, pp.79-83. (The authors found that formulae 8 on p.81, 2.268, and 5 on p.83 are faulty).

5. Luré, A.I. — Three-dimensional Problems of the Theory of Elasticity, Interscience, New York, 1964, pp.350-370.

6. Magnus, W. and Oberhettinger, F. — Functions of Mathematical Physics, Chelsea Publishing Company, New York, 1954, pp.51, 62.

7. Morse, P.M. and Feshbach, H. — Methods of Theoretical Physics, McGraw-Hill, New York, 1953, pp.782, 1325.

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